

Tight lower bounds for the complexity of **multicoloring***

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Abstract

In the *multicoloring* problem, also known as $(a:b)$ -coloring or b -fold coloring, we are given a graph G and a set of a colors, and the task is to assign a subset of b colors to each vertex of G so that adjacent vertices receive disjoint color subsets. This natural generalization of the classic coloring problem (the $b = 1$ case) is equivalent to finding a homomorphism to the Kneser graph $KG_{a,b}$, and gives relaxations approaching the fractional chromatic number.

We study the complexity of determining whether a graph has an $(a:b)$ -coloring. Our main result is that this problem does not admit an algorithm with running time $f(b) \cdot 2^{o(\log b) \cdot n}$, for any computable $f(b)$, unless the Exponential Time Hypothesis (ETH) fails. A $(b+1)^n \cdot \text{poly}(n)$ -time algorithm due to Nederlof [2008] shows that this is tight. A direct corollary of our result is that the graph homomorphism problem does not admit a $2^{O(n+h)}$ algorithm unless ETH fails, even if the target graph is required to be a Kneser graph. This refines the understanding given by the recent lower bound of Cygan et al. [SODA 2016].

The crucial ingredient in our hardness reduction is the usage of *detecting matrices* of Lindström [Canad. Math. Bull., 1965], which is a combinatorial tool that, to the best of our knowledge, has not yet been used for proving complexity lower bounds. As a side result, we prove that the running time of the algorithms of Abasi et al. [MFCS 2014] and of Gabizon et al. [ESA 2015] for the r -monomial detection problem are optimal under ETH.

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1 Introduction

The complexity of determining the chromatic number of a graph is undoubtedly among the most intensively studied computational problems. Countless variants, extensions, and generalizations of graph colorings have been introduced and investigated. Here, we focus on *multicolorings*, also known as $(a:b)$ -colorings. In this setting, we are given a graph G , a palette of a colors, and a number $b \leq a$. An $(a:b)$ -coloring of G is any assignment of b distinct colors to each vertex so that adjacent vertices receive disjoint subsets of colors. The $(a:b)$ -COLORING problem asks whether G admits an $(a:b)$ -coloring. Note that for $b = 1$ we obtain the classic graph coloring problem. The smallest a for which an $(a:b)$ -coloring exists, is called the *b-fold chromatic number*, denoted by $\chi_b(G)$.

The motivation behind $(a:b)$ -colorings can be perhaps best explained by showing the connection with the *fractional chromatic number*. The fractional chromatic number of a graph G , denoted $\chi_f(G)$, is the optimum value of the natural LP relaxation of the problem of computing the chromatic number of G , expressed as finding a cover of the vertex set using the minimum possible number of independent sets. It can be easily seen that by relaxing the standard coloring problem by allowing b times more colors while requiring that every vertex receives b colors and adjacent vertices receive disjoint subsets, with increasing b we approximate the fractional chromatic number better and better. Consequently, $\lim_{b \rightarrow \infty} \chi_b(G)/b = \chi_f(G)$.

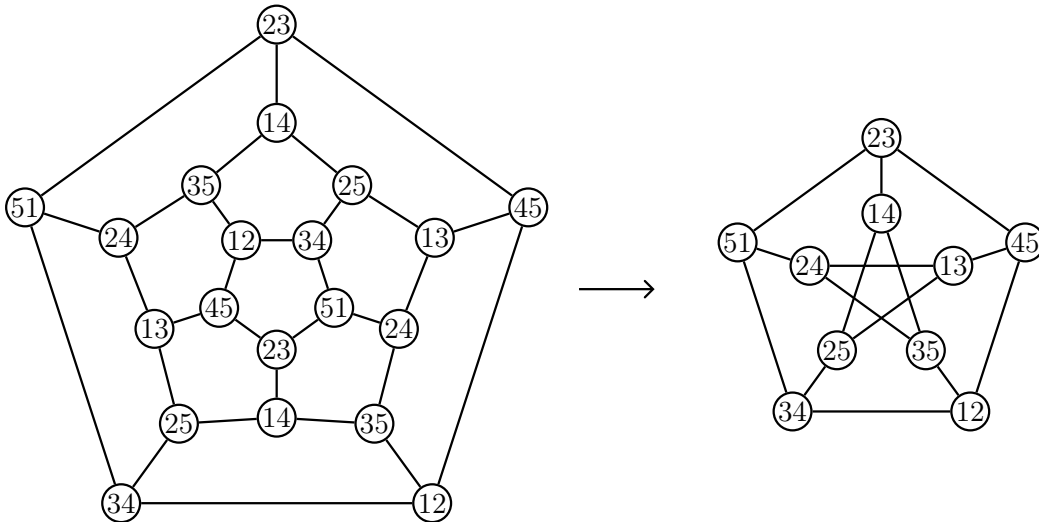


Figure 1: A $(5:2)$ -coloring of the dodecahedron (left) which can be seen as a homomorphism to $KG_{5,2}$ (the Petersen graph, right). The homomorphism is given by identifying the pairs of opposite vertices in the corresponding regular solid.

Another interesting connection concerns *Kneser graphs*. Recall that for positive integers a, b with $b < a/2$, the Kneser graph $KG_{a,b}$ has all b -element subsets of $\{1, 2, \dots, a\}$ as vertices, and two subsets are considered adjacent if and only if they are disjoint. For instance, $KG_{5,2}$ is the well-known Petersen graph (see Fig. 1, right). Thus, $(a:b)$ -coloring of a graph G can be interpreted as a homomorphism from G to the Kneser graph $KG_{a,b}$ (see Fig. 1). Kneser graphs are well studied in the context of graph colorings mostly due to the celebrated result of Lovász [29], who determined their chromatic number, initiating the field of topological combinatorics.

Multicolorings and $(a:b)$ -colorings have been studied both from combinatorial [7, 12, 27] and algorithmic [5, 19, 20, 25, 26, 30, 31, 34] points of view. The main real-life motivation comes from

the problem of assigning frequencies to nodes in a cellular network so that adjacent nodes receive disjoint sets of frequencies on which they can operate. This makes (near-)planar and distributed settings particularly interesting for practical applications. We refer to the survey of Halldórsson and Kortsarz [18] for a broader discussion.

In this paper we focus on the paradigm of exact exponential time algorithms: given a graph G on n vertices and numbers $a \geq b$, we would like to determine whether G is $(a:b)$ -colorable as quickly as possible. Since the problem is already NP-hard for $a = 3$ and $b = 1$, we do not expect it to be solvable in polynomial time, and hence look for an efficient exponential-time algorithm. A straightforward dynamic programming approach yields an algorithm with running time¹ $\mathcal{O}^*(2^n \cdot (b+1)^n)$ as follows. For each function $\eta: V(G) \rightarrow \{0, 1, \dots, b\}$ and each $k = 0, 1, \dots, a$, we create one boolean entry $D[\eta, k]$ denoting whether one can choose k independent sets in G so that every vertex $v \in V(G)$ is covered exactly $\eta(v)$ times. Then value $D[\eta, k]$ can be computed as a disjunction of values $D[\eta', k-1]$ over η' obtained from η by subtracting 1 on vertices from some independent set in G .

This simple algorithm can be improved by finding an appropriate algebraic formula for the number of $(a:b)$ -colorings of the graph and using the inclusion-exclusion principle to compute it quickly, similarly as in the case of standard colorings [2]. Such an algebraic formula was given by Nederlof [33, Theorem 3.5] in the context of a more general MULTI SET COVER problem. Nederlof also observed that in the case of $(a:b)$ -COLORING, a simple application of the inclusion-exclusion principle to compute the formula yields an $\mathcal{O}^*((b+1)^n)$ -time exponential-space algorithm. Hua et al. [22] noted that the formulation of Nederlof [33] for MULTI SET COVER can be also used to obtain a polynomial-space algorithm for this problem. By taking all maximal independent sets to be the family in the MULTI SET COVER problem, and applying the classic Moon-Moser upper bound on their number [32], we obtain an algorithm for $(a:b)$ -COLORING that runs in time $\mathcal{O}^*(3^{n/3} \cdot (b+1)^n)$ and uses polynomial space. Note that by plugging $b = 1$ to the results above, we obtain algorithms for the standard coloring problem with running time $\mathcal{O}^*(2^n)$ and exponential space usage, and with running time $\mathcal{O}^*(2.8845^n)$ and polynomial space usage, which almost matches the fastest known procedures [2].

The complexity of $(a:b)$ -COLORING becomes particularly interesting in the context of the GRAPH HOMOMORPHISM problem: given graphs G and H , with n and h vertices respectively, determine whether G admits a homomorphism to H . By the celebrated result of Hell and Nešetřil [21] the problem is in P if H is bipartite and NP-complete otherwise. For quite a while it was open whether there is an algorithm for GRAPH HOMOMORPHISM running in time $2^{\mathcal{O}(n+h)}$. It was recently answered in the negative by Cygan et al. [9]; more precisely, they proved that an algorithm with running time $2^{o(n \log h)}$ would contradict the Exponential Time Hypothesis (ETH) of Impagliazzo et al. [23]. However, GRAPH HOMOMORPHISM is a very general problem, hence researchers try to uncover a more fine-grained picture and identify families of graphs \mathcal{H} such that the problem can be solved more efficiently whenever $H \in \mathcal{H}$. For example, Fomin, Heggernes and Kratsch [13] showed that when H is of treewidth at most t , then GRAPH HOMOMORPHISM can be solved in time $\mathcal{O}^*((t+3)^n)$. It was later extended to graphs of cliquewidth bounded by t , with an $\mathcal{O}^*((2t+1)^{\max\{n,h\}})$ time bound by Wahlström [36]. On the other hand, H needs not be sparse to admit efficient homomorphism testing: the family of cliques admits the $\mathcal{O}^*(2^n)$ running time as shown by Björklund et al. [2]. As noted above, this generalizes to Kneser graphs $KG_{a,b}$, by the $\mathcal{O}^*((b+1)^n)$ -time algorithm of Nederlof. In this context, the natural question is whether the appearance of b in the base of the exponent is necessary, or is there an algorithm running in time $\mathcal{O}^*(c^n)$ for some universal constant c independent of b .

¹The $\mathcal{O}^*(\cdot)$ notation hides factors polynomial in the input size.

Our contribution. We show that the algorithms for $(a:b)$ -COLORING mentioned above are essentially optimal under the Exponential Time Hypothesis. Specifically, we prove the following results:

Theorem 1. *If there is an algorithm for $(a:b)$ -COLORING that runs in time $f(b) \cdot 2^{o(\log b) \cdot n}$, for some computable function $f(b)$, then ETH fails. This holds even if the algorithm is only required to work on instances where $a = \Theta(b^2 \log b)$.*

Corollary 2. *If there is an algorithm for GRAPH HOMOMORPHISM that runs in time $f(h) \cdot 2^{o(\log \log h) \cdot n}$, for some computable function $f(h)$, then ETH fails. This holds even if the algorithm is only required to work on instances where H is a Kneser graph $KG_{a,b}$ with $a = \Theta(b^2 \log b)$.*

The bound for $(a:b)$ -COLORING is tight, as the straightforward $\mathcal{O}^*(2^n \cdot (b+1)^n) = 2^{\mathcal{O}(\log b) \cdot n}$ dynamic programming algorithm already shows. At first glance, one might have suspected that $(a:b)$ -COLORING, as an interpolation between classical coloring and fractional coloring, both solvable in $2^{\mathcal{O}(n)}$ time [17], should be just as easy; Theorem 1 refutes this suspicion.

Corollary 2 in particular excludes any algorithm for testing homomorphisms into Kneser graphs with running time $2^{\mathcal{O}(n+h)}$. It cannot give a tight lower bound matching the result of Cygan et al. [9] for general homomorphisms, because $h = |V(KG_{a,b})| = \binom{a}{b}$ is not polynomial in b . On the other hand, it exhibits the first explicit family of graphs H for which the complexity of GRAPH HOMOMORPHISM increases with h .

In our proof, we first show a lower bound for the list variant of the problem, where every vertex is given a list of colors that can be assigned to it (see Section 2 for formal definitions). The list version is reduced to the standard version by introducing a large Kneser graph $KG_{a+b,b}$; we need a and b to be really small so that the size of this Kneser graph does not dwarf the size of the rest of the construction. However, this is not necessary for the list version, where we obtain lower bounds for a much wider range of functions $b(n)$.

Theorem 3. *If there is an algorithm for LIST $(a:b)$ -COLORING that runs in time $2^{o(\log b) \cdot n}$, then ETH fails. This holds even if the algorithm is only required to work on instances where $a = \Theta(b^2 \log b)$ and $b = \Theta(b(n))$ for an arbitrarily chosen polynomial-time computable function $b(n)$ such that $b(n) \in \omega(1)$ and $b(n) = \mathcal{O}(n/\log n)$.*

The crucial ingredient in the proof of Theorem 3 is the usage of d -detecting matrices introduced by Lindström [28]. We choose to work with their combinatorial formulation, hence we shall talk about d -detecting families. Suppose we are given some universe U and there is an unknown function $f: U \rightarrow \{0, 1, \dots, d-1\}$, for some fixed positive integer d . One may think of U as consisting of coins of unknown weights that are integers between 0 and $d-1$. We would like to learn f (the weight of every coin) by asking a small number of queries of the following form: for a subset $X \subseteq U$, what is $\sum_{e \in X} f(e)$ (the total weight of coins in X)? A set of queries sufficient for determining all the values of an arbitrary f is called a d -detecting family. Of course f can be learned by asking $|U|$ questions about single coins, but it turns out that significantly fewer questions are needed: there is a d -detecting family of size $\mathcal{O}(|U|/\log |U|)$, for every fixed d [28]. The logarithmic factor in the denominator will be crucial for deriving our lower bound.

Let us now sketch how d -detecting families are used in the proof of Theorem 3. Given an instance φ of 3-SAT with n variables and $\mathcal{O}(n)$ clauses, and a number $b \leq n/\log n$, we will construct an instance G of LIST $(a:b)$ -COLORING for some a . This instance will have a positive answer if and only if φ is satisfiable, and the constructed graph G will have $\mathcal{O}(n/\log b)$ vertices. It can be easily seen that this will yield the promised lower bound.

Partition the clause set C of φ into groups C_1, C_2, \dots, C_p , each of size roughly b ; thus $p = \mathcal{O}(n/b)$. Similarly, partition the variable set V of φ into groups V_1, \dots, V_q , each of size roughly $\log_2 b$; thus

$q = \mathcal{O}(n/\log b)$. In the output instance we create one vertex per each variable group—hence we have $\mathcal{O}(n/\log b)$ such vertices—and one block of vertices per each clause group, whose size will be determined in a moment. Our construction ensures that the set of colors assigned to a vertex created for a variable group misses one color from some subset of b colors. The choice of the missing color corresponds to one of $2^{\log_2 b} = b$ possible boolean assignments to the variables of the group.

Take any vertex u from a block of vertices created for some clause group C_j . We make it adjacent to vertices constructed for precisely those variable groups V_i , for which there is some variable in V_i that occurs in some clause of C_j . This way, u can only take a subset of the above missing colors corresponding to the chosen assignment on variables relevant to C_j . By carefully selecting the list of u , and some additional technical gadgeteering, we can express a constraint of the following form: the total number of satisfied literals in some subset of clauses of C_j is exactly some number. Thus, we could verify that every clause of C_j is satisfied by creating a block of $|C_j|$ vertices, each checking one clause. However, the whole graph output by the reduction would then have $\mathcal{O}(n)$ vertices, and we would not obtain any non-trivial lower bound. Instead, we create one vertex per each question in a d -detecting family on the universe $U = C_j$, which has size $\mathcal{O}(|C_j|/\log |C_j|) = \mathcal{O}(|C_j|/\log b)$. Then, the total number of vertices in the constructed graph will be $\mathcal{O}(n/\log b)$, as intended.

Finally, we observe that from our main result one can infer a lower bound for the complexity of the (r, k) -MONOMIAL TESTING problem. Recall that in this problem we are given an arithmetic circuit that evaluates a homogenous polynomial $P(x_1, x_2, \dots, x_n)$ over some field \mathbb{F} ; here, a polynomial is homogenous if all its monomials have the same total degree k . The task is to verify whether P has some monomial in which every variable has individual degree not larger than r , for a given parameter r . Abasi et al. [1] gave a randomized algorithm solving this problem in time $\mathcal{O}^*(2^{\mathcal{O}(k \cdot \frac{\log r}{r})})$, where k is the degree of the polynomial, assuming that $\mathbb{F} = \text{GF}(p)$ for a prime $p \leq 2r^2 + 2r$. This algorithm was later derandomized by Gabizon et al. [14] within the same running time, but under the assumption that the circuit is *non-cancelling*: it has only input, addition, and multiplication gates. Abasi et al. [1] and Gabizon et al. [14] gave a number of applications of low-degree monomial detection to concrete problems. For instance, r -SIMPLE k -PATH, the problem of finding a walk of length k that visits every vertex at most r times, can be solved in time $\mathcal{O}^*(2^{\mathcal{O}(k \cdot \frac{\log r}{r})})$. However, for r -SIMPLE k -PATH, as well as other problems that can be tackled using this technique, the best known lower bounds under ETH exclude only algorithms with running time $\mathcal{O}^*(2^{\mathcal{O}(\frac{k}{r})})$. Whether the $\log r$ factor in the exponent is necessary was left open by Abasi et al. and Gabizon et al.

We observe that the LIST $(a:b)$ -COLORING problem can be reduced to (r, k) -MONOMIAL TESTING over the field $\text{GF}(2)$ in such a way that an $\mathcal{O}^*(2^{k \cdot \mathcal{O}(\frac{\log r}{r})})$ -time algorithm for the latter would imply a $2^{o(\log b) \cdot n}$ -time algorithm for the former, which would contradict ETH. Thus, we show that the known algorithms for (r, k) -MONOMIAL TESTING most probably cannot be sped up in general; nevertheless, the question of lower bounds for specific applications remains open. However, going through LIST $(a:b)$ -COLORING to establish a lower bound for (r, k) -MONOMIAL TESTING is actually quite a detour, because the latter problem has a much larger expressive power. Therefore, we also give a more straightforward reduction that starts from a convenient form of SUBSET SUM; this reduction also proves the lower bound for a wider range of r , expressed as a function of k .

Outline. In Section 2 we set up the notation as well as recall definitions and well-known facts. We also discuss d -detecting families, the main combinatorial tool used in our reduction. In Section 3 we prove the lower bound for the list version of the problem, i.e., Theorem 3. In Section 4 we give a reduction from the list version to the standard version, thereby proving Theorem 1. Section 5 is devoted to deriving lower bounds for low-degree monomial testing.

2 Preliminaries

Notation. We use standard graph notation, see e.g. [10, 11]. All graphs we consider in this paper are simple and undirected. For an integer k , we denote $[k] = \{0, \dots, k-1\}$. By \uplus we denote the disjoint union, i.e., by $A \uplus B$ we mean $A \cup B$ with the indication that A and B are disjoint. If I and J are instances of decision problems P and R , respectively, then we say that I and J are *equivalent* if either both I and J are YES-instances of respective problems, or both are NO-instances.

Exponential-Time Hypothesis. The Exponential Time Hypothesis (ETH) of Impagliazzo et al. [23] states that there exists a constant $c > 0$, such that there is no algorithm solving 3-SAT in time $\mathcal{O}^*(2^{cn})$. During the recent years, ETH became the central conjecture used for proving tight bounds on the complexity of various problems. One of the most important results connected to ETH is the *Sparsification Lemma* [24], which essentially gives a reduction from an arbitrary instance of k -SAT to an instance where the number of clauses is linear in the number of variables. The following well-known corollary can be derived by combining ETH with the Sparsification Lemma.

Theorem 4 (see e.g. Theorem 14.4 in [10]). *Unless ETH fails, there is no algorithm for 3-SAT that runs in time $2^{o(n+m)}$, where n, m denote the numbers of variables and clauses, respectively.*

We need the following regularization result of Tovey [35]. Following Tovey, by (3,4)-SAT we call the variant of 3-SAT where each clause of the input formula contains exactly 3 different variables, and each variable occurs in at most 4 clauses.

Lemma 5 ([35]). *Given a 3-SAT formula φ with n variables and m clauses one can transform it in polynomial time into an equivalent (3,4)-SAT instance φ' with $\mathcal{O}(n+m)$ variables and clauses.*

Corollary 6. *Unless ETH fails, there is no algorithm for (3,4)-SAT that runs in time $2^{o(n)}$, where n denotes the number of variables of the input formula.*

List and nonuniform list $(a:b)$ -coloring For integers a, b and a graph G with a function $L: V(G) \rightarrow 2^{[a]}$ (assigning a list of colors to every vertex), an L - $(a:b)$ -coloring of G is an assignment of exactly b colors from $L(v)$ to each vertex $v \in V(G)$, such that adjacent vertices get disjoint color sets. The LIST $(a:b)$ -COLORING problem asks, given (G, L) , whether an L - $(a:b)$ -coloring of G exists.

As an intermediary step of our reduction, we will use the following generalization of list colorings where the number of demanded colors varies with every vertex. For integers a, b , a graph G with a function $L: V(G) \rightarrow 2^{[a]}$ and a *demand function* $\beta: V(G) \rightarrow \{1, \dots, b\}$, an L - $(a:\beta)$ -coloring of G is an assignment of exactly $\beta(v)$ colors from $L(v)$ to each vertex $v \in V(G)$, such that adjacent vertices get disjoint color sets. NONUNIFORM LIST $(a:b)$ -COLORING is then the problem in which given (G, L, β) we ask if an L - $(a:\beta)$ -coloring of G exists.

d -detecting families. In our reductions the following notion plays a crucial role.

Definition 7. *A d -detecting family for a finite set U is a family $\mathcal{F} \subseteq 2^U$ of subsets of U such that for every two functions $f, g: U \rightarrow \{0, \dots, d-1\}$, $f \neq g$, there is a set S in the family such that $\sum_{x \in S} f(x) \neq \sum_{x \in S} g(x)$.*

A deterministic construction of sublinear, d -detecting families was given by Lindström [28], together with a proof that even the constant factor 2 in the family size cannot be improved.

Theorem 8 ([28]). *For every constant $d \in \mathbb{N}$ and finite set U , there is a d -detecting family \mathcal{F} on U of size $\frac{2|U|}{\log_d |U|} \cdot (1 + o(1))$. Furthermore, \mathcal{F} can be constructed in time polynomial in $|U|$.*

Other constructions, generalizations, and discussion of similar results can be found in Grebinski and Kucherov [16], and in Bshouty [3]. Note that the expression $\sum_{x \in S} f(x)$ is just the product of f as a vector in $[d]^{|U|}$ with the characteristic vector of S . Hence, instead of subset families, Lindström speaks of *detecting vectors*, while later works see them as *detecting matrices*, that is, $(0, 1)$ -matrices with these vectors as rows (which define an injection on $[d]^{|U|}$ despite having few rows). Similar definitions appear in the study of query complexity, e.g., as in the popular Mastermind game [6].

While known polynomial deterministic constructions of detecting families involve some number theory or fourier analysis, their existence can be argued with an elementary probabilistic argument. Intuitively, a random subset $S \subseteq U$ will distinguish two distinct functions $f, g : U \rightarrow \{0, \dots, d-1\}$ (meaning $\sum_{x \in S} f(x) \neq \sum_{x \in S} g(x)$) with probability at least $\frac{1}{2}$. This is because any x where f and g disagree is taken or not taken into S with probability $\frac{1}{2}$, while sums over S cannot agree in both cases simultaneously, as they differ by $f(x)$ and $g(x)$ respectively. There are $d^n \cdot d^n$ function pairs to be distinguished. In any subset of pairs, at least half are distinguished by a random set in expectation, thus at least one such set exists. Repeatedly finding such a set for undistinguished pairs, we get $|\log_{\frac{1}{2}}(d^n \cdot d^n)| = \mathcal{O}(n \log d)$ sets that distinguish all functions. More strongly though, when two functions differ on more values, the probability of distinguishing them increases significantly. Hence we need fewer random sets to distinguish all pairs of distant functions. On the other hand, there are few function pairs that are close, so we need few random sets to distinguish them all as well. This allows to show that in fact $\mathcal{O}(\frac{n}{\log_d n})$ random sets are enough to form a d -detecting family with positive probability [16].

3 Hardness of LIST $(a:b)$ -COLORING

In this section we show our main technical contribution: an ETH-based lower bound for LIST $(a:b)$ -COLORING. The key part is reducing an n -variable instance 3-SAT to an instance of NONUNIFORM LIST $(a:b)$ -COLORING with only $\mathcal{O}(\frac{n}{\log b})$ vertices. Next, it is rather easy to reduce NONUNIFORM LIST $(a:b)$ -COLORING to LIST $(a:b)$ -COLORING. We proceed with the first, key part.

3.1 The nonuniform case

We prove the following theorem through the remaining part of this section.

Theorem 9. *For any instance ϕ of (3,4)-SAT with n variables and any integer $2 \leq b \leq n/\log_2 n$, there is an equivalent instance (G, β, L) of NONUNIFORM LIST $(a:2b)$ -COLORING such that $a = \mathcal{O}(b^2 \log b)$, $|V(G)| = \mathcal{O}(\frac{n}{\log b})$ and G is 3-colorable. Moreover, the instance (G, β, L) and the 3-coloring of G can be constructed in $\text{poly}(n)$ time.*

Consider an instance ϕ of 3-SAT where each variable appears in at most four clauses. Let V be the set of its variables and C be the set of its clauses. Note that $\frac{1}{3}|V| \leq |C| \leq \frac{4}{3}|V|$. Let $a = 12b^2 \cdot \lfloor \log_2 b \rfloor$. We shall construct, for some integers $n_V = \mathcal{O}(|V|/\log b)$ and $n_C = \mathcal{O}(|C|/b)$:

- a partition $V = V_1 \uplus \dots \uplus V_{n_V}$ of variables into groups of size at most $\lfloor \log_2 b \rfloor$,
- a partition $C = C_1 \uplus \dots \uplus C_{n_C}$ of clauses into groups of size at most b ,
- a function $\sigma : \{1, \dots, n_V\} \rightarrow [12 \cdot b \cdot \lfloor \log_2 b \rfloor]$,

such that the following condition holds:

For any $j = 1, \dots, n_C$, the variables occurring in clauses of C_j are all different and they all belong to pairwise different variable groups. Moreover, the indices of these groups are mapped to pairwise different values by σ . (✕)

In other words, any two literals of clauses in C_j have different variables, and if they belong to V_i and $V_{i'}$ respectively, then $\sigma(i) \neq \sigma(i')$.

Lemma 10. *Partitions $V = V_1 \uplus \dots \uplus V_{n_V}$, $C = C_1 \uplus \dots \uplus C_{n_C}$ and a function σ satisfying (\boxtimes) can be found in time $\mathcal{O}(n)$.*

Proof. We first group variables, in a way such that the following holds: (P1) the variables occurring in any clause are different and belong to different variable groups. To this end, consider the graph G_1 with variables as vertices and edges between any two variables that occur in a common clause (i.e. the primal graph of ϕ). Since no clause contains repeated variables, G_1 has no loops. Since every variable of ϕ occurs in at most four clauses, and since those clauses contain at most two other variables, the maximum degrees of G_1 is at most 8. Hence G_1 can be greedily colored with 9 colors. Then, we refine the partition given by colors to make every group have size at most $\lfloor \log_2 b \rfloor$, producing in total at most $n_V := \lceil |V| / \lfloor \log_2 b \rfloor \rceil + 9$ groups V_1, \dots, V_{n_V} . (P1) holds, because any two variables occurring in a common clause are adjacent in G_1 , and thus get different colors, and thus are assigned to different groups.

Next, we group clauses in a way such that: (P2) the variables occurring in clauses of a group C_j are all different and belong to different variable groups. For this, consider the graph G_2 with clauses as vertices, and with an edge between clauses if they contain two different variables from the same variable group. By (P1), G_2 has no loops. Since every clause contains exactly 3 variables, each variable is in a group with at most $\lfloor \log_2 b \rfloor - 1$ others, and every such variable occurs in at most 4 clauses, the maximum degree of G_2 is at most $12(\lfloor \log_2 b \rfloor - 1)$. We can therefore color G_2 greedily with $12\lfloor \log_2 b \rfloor$ colors. Similarly as before, we partition clauses into $n_C := \lceil |C|/b \rceil + 12\lfloor \log_2 b \rfloor$ monochromatic groups C_1, \dots, C_{n_C} of size at most b each. Then (P2) holds by construction of the coloring.

Finally, consider a graph G_3 with variable groups as vertices, and with an edge between two variable groups if they contain two different variables occurring in clauses from a common clause group. More precisely, V_i and $V_{i'}$ are adjacent if there are two different variables $x \in V_i$ and $x' \in V_{i'}$, and a clause group C_j with clauses c and c' (possibly $c = c'$), such that x occurs in c and x' occurs in c' . By (P2), G_3 has no loops. Since a variable has at most $\lfloor \log_2 b \rfloor - 1$ other variables in its group, each of these variables occur in at most 4 clauses, each of these clauses has at most $b - 1$ other clauses in its group, and each of these contains exactly 3 variables, the maximum degree of G_3 is at most $4 \cdot (\lfloor \log_2 b \rfloor - 1) \cdot (b - 1) \cdot 3$. We can therefore color it greedily into $12b\lfloor \log_2 b \rfloor$ colors. Let σ be the resulting coloring. By (P2) and the construction of this coloring, (\boxtimes) holds.

The colorings can be found in linear time using standard techniques. Note that we have $n_V = \lceil |V| / \lfloor \log_2 b \rfloor \rceil + 9 = \mathcal{O}(|V| / \log b)$. Moreover, since $b \leq n / \log_2 n$, we get $\log_2 b \leq \log_2 n \leq \frac{n}{b} = \Theta(|C|/b)$ and hence $n_C = \lceil |C|/b \rceil + 12\lfloor \log_2 b \rfloor = \mathcal{O}(|C|/b)$. \square

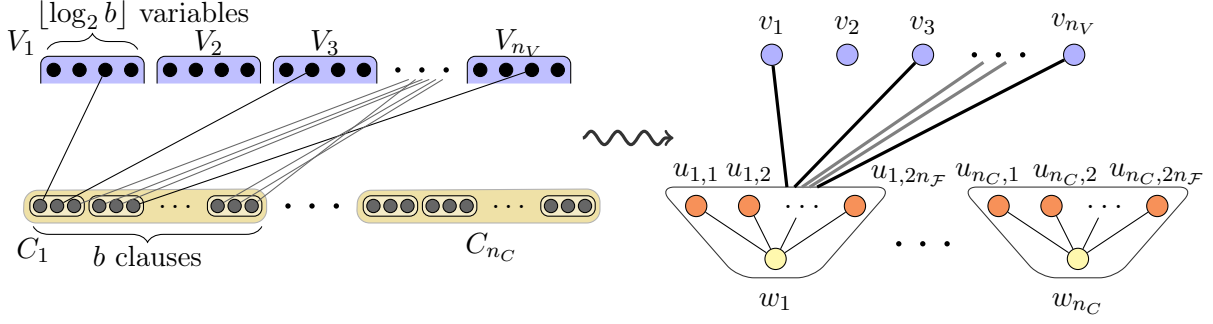


Figure 2: (left) The groups of variables and clauses of the formula; literals in C_1 are joined with their variables. Since no variable of V_2 occurs in C_1 , we have $2 \notin I_1$ – this may allow us to make $\sigma(2)$ the same number as $\sigma(3)$, say, reducing the total number a of colors needed. (right) The constructed graph; thick lines represent edges to all vertices corresponding to C_1 .

For every $1 \leq i \leq n_V$, the set V_i of variables admits $2^{|V_i|} \leq b$ different assignments. We will therefore say that each assignment on V_i is *given* by an integer $x \in [b]$, for example by interpreting the first $|V_i|$ bits of the binary representation of x as truth values for variables in V_i . Note that when $|V_i| < \log_2 b$, different integers from $[b]$ may give the same assignment on V_i .

For $1 \leq j \leq n_C$, let $I_j \subseteq \{1, \dots, n_V\}$ be the set of indices of variable groups that contain some variable occurring in the clauses of C_j . Since every clause contains exactly three literals, property (\boxtimes) means that $|I_j| = 3|C_j|$ and that σ is injective over each I_j . See Fig. 2.

For $1 \leq j \leq n_C$, let $\{C_{j,1}, \dots, C_{j,n_F}\}$ be a 4-detecting family of subsets of C_j , for some $n_F = \mathcal{O}(\frac{b}{\log b})$ (we can assume n_F does not depend on j by adding arbitrary sets when $|C_j| < b$). For every $1 \leq k \leq n_F$, let $C_{j,n_F+k} = C_j \setminus C_{j,k}$.

We are now ready to build the graph G , the demand function $\beta : V(G) \rightarrow \{1, \dots, 2b\}$, and the list assignment L as follows.

- (1) For $1 \leq i \leq n_V$, create a vertex v_i with $\beta(v_i) = b - 1$ and $L(v_i) = \{b \cdot \sigma(i) + x \mid x \in [b]\}$.
- (2) For $1 \leq j \leq n_C$ and $1 \leq k \leq 2n_F$, create a vertex $u_{j,k}$ adjacent to each v_i for $i \in I_j$.
Let $\beta(u_{j,k}) = |C_{j,k}|$ and

$$L(u_{j,k}) = \{b \cdot \sigma(i) + x \mid 1 \leq i \leq n_V, x \in [2^{|V_i|}] \text{ such that } x \text{ gives an assignment of } V_i \text{ that satisfies some clause of } C_{j,k}\}.$$

- (3) For $1 \leq j \leq n_C$, create a vertex w_j , adjacent to each v_i for $i \in I_j$ and to each $u_{j,k}$ ($1 \leq k \leq 2n_F$).
Let $\beta(w_j) = 2|C_j|$ and $L(w_j) = \bigcup_{i \in I_j} \{b \cdot \sigma(i) + x \mid x \in [b]\}$.

Before giving a detailed proof of the correctness, let us describe the reduction in intuitive terms. Note that vertices of type v_i get all but one color from their list; this missing color, say $b \cdot \sigma(i) + x_i$, for some $x_i \in [b]$, defines an assignment on V_i . For every $j = 1, \dots, n_C$ the goal of the gadget consisting of w_j and vertices $u_{j,k}$ is to express the constraint that every clause in C_j has a literal satisfied by this assignment. Since $w_j, u_{j,k}$ are adjacent to all vertices in $\{v_i \mid i \in I_j\}$, they may only use the missing colors (of the form $b \cdot \sigma(i) + x_i$, where $i \in I_j$). Since $|I_j| = 3|C_j|$, there are $3|C_j|$ such colors and $2|C_j|$ of them go to w_j . This leaves exactly $|C_j|$ colors for vertices of type $u_{j,k}$, corresponding to a choice of $|C_j|$ satisfied literals from the $3|C_j|$ literals in clauses of C_j . The lists and demands for vertices $u_{j,k}$ guarantee that exactly $|C_{j,k}|$ chosen satisfied literals occur in clauses of $C_{j,k}$. The properties of 4-detecting families will ensure that every clause has exactly one chosen, satisfied literal, and hence at least one satisfied literal. We proceed with formal proofs.

Lemma 11. *If ϕ is satisfiable then G is $L(a:\beta)$ -colorable.*

Proof. Consider a satisfying assignment η for ϕ . For $1 \leq i \leq n_V$, let $x_i \in [2^{|V_i|}]$ be an integer giving the same assignment on V_i as η . For every clause c of ϕ , choose one literal satisfied by η in it, and let i_c be index of the group V_{i_c} containing the literal's variable. Let $\alpha : V(G) \rightarrow \binom{[a]}{\leq 2b}$ be the $L(a:\beta)$ -coloring of G defined as follows, for $1 \leq i \leq n_V$, $1 \leq j \leq n_C$, $1 \leq k \leq 2n_{\mathcal{F}}$:

- $\alpha(v_i) = L(v_i) \setminus \{b \cdot \sigma(i) + x_i\}$
- $\alpha(u_{j,k}) = \{b \cdot \sigma(i_c) + x_{i_c} \mid c \in C_{j,k}\}$
- $\alpha(w_j) = \{b \cdot \sigma(i) + x_i \mid i \in I_j \setminus \{i_c \mid c \in C_j\}\}$.

Let us first check that every vertex v gets colors from its list $L(v)$ only. This is immediate for vertices v_i and w_j , while for $u_{j,k}$ it follows from the fact that x_{i_c} gives a partial assignment to V_i that satisfies some clause of $C_{j,k}$.

Now let us check that for every vertex v , the coloring α assigns exactly $\beta(v)$ colors to v . For $\alpha(v_i)$ this follows from the fact that $|L(v_i)| = b$ and $0 \leq x_i < 2^{|V_i|} \leq b$. Since by property (\boxtimes) , σ is injective on I_j , and thus on $\{i_c \mid c \in C_{j,k}\} \subseteq I_j$, we have $|\alpha(u_{j,k})| = |C_{j,k}| = b(u_{j,k})$. Similarly, since σ is injective on I_j and $|I_j \setminus \{i_c \mid c \in C_j\}| = 3|C_j| - |C_j| = 2|C_j|$, we get $|\alpha(w_j)| = 2|C_j| = \beta(w_j)$.

It remains to argue that the sets assigned to any two adjacent vertices are disjoint. There are three types of edges in the graph, namely $v_i u_{j,k}$, $v_i w_j$, and $w_j u_{j,k}$. The disjointness of $\alpha(w_j)$ and $\alpha(u_{j,k})$ is immediate from the definition of α , since $C_{j,k} \subseteq C_j$. Fix $j = 1, \dots, n_C$. Since σ is injective on I_j , for any two different $i, i' \in I_j$, we have $b \cdot \sigma(i) + x_i \notin L(v_{i'})$. Hence,

$$\bigcup_{i \in I_j} \alpha(v_i) = \{b \cdot \sigma(i) + x \mid i \in I_j \text{ and } x \in [b]\} \setminus \{b \cdot \sigma(i) + x_i \mid i \in I_j\}.$$

Since $\alpha(u_{j,k}), \alpha(w_j) \subseteq \{b \cdot \sigma(i) + x_i \mid i \in I_j\}$, it follows that edges of types $v_i u_{j,k}$ and $v_i w_j$ received disjoint sets of colors on their endpoints, concluding the proof. \square

Lemma 12. *If G is $L(a:\beta)$ -colorable then ϕ is satisfiable.*

Proof. Assume that G is $L(a:\beta)$ -colorable, and let α be the corresponding coloring.

For $1 \leq i \leq n_V$, we have $|L(v_i)| = b$ and $|\alpha(v_i)| = b - 1$, so v_i misses exactly one color from its list. Let $b \cdot \sigma(i) + x_i$, for some $x_i \in [b]$, be the missing color. We want to argue that the assignment x for ϕ given by x_i on each V_i satisfies ϕ .

Consider any clause group C_j , for $1 \leq j \leq n_C$. Every vertex in $\{w_j\} \cup \{u_{j,k} \mid 1 \leq k \leq 2n_{\mathcal{F}}\}$ contains $\{v_i \mid i \in I_j\}$ in its neighborhood. Therefore, the sets $\alpha(u_{j,k})$ and $\alpha(w_j)$ are disjoint from $\bigcup_{i \in I_j} \alpha(v_i)$. Since $L(u_{j,k}), L(w_j) \subseteq \{b \cdot \sigma(i) + x' \mid i \in I_j, x' \in [b]\}$, we get that $\alpha(u_{j,k})$ and $\alpha(w_j)$ are contained in the set of missing colors $\{b \cdot \sigma(i) + x_i \mid i \in I_j\}$ (corresponding to the chosen assignment). By property (\boxtimes) , this set has exactly $|I_j| = 3|C_j|$ different colors. Of these, exactly $2|C_j|$ are contained in $\alpha(w_j)$. Let the remaining $|C_j|$ colors be $\{b \cdot \sigma(i) + x_i \mid i \in J_j\}$, for some subset $J_j \subseteq I_j$ of $|C_j|$ indices.

Since $\alpha(u_{j,k})$ is disjoint from $\alpha(w_j)$, we have $\alpha(u_{j,k}) \subseteq \{b \cdot \sigma(i) + x_i \mid i \in J_j\}$ for all k . By definition of I_j , for every $i \in J_j \subseteq I_j$ there is a variable in V_i that appears in some clause of C_j . By property (\boxtimes) , it can only occur in one such clause, so let l_i be the literal in the clause of C_j where it appears. For every color $b \cdot \sigma(i) + x_i \in \alpha(u_{j,k})$, by definition of the lists for $u_{j,k}$ we know that x_i gives a partial assignment to V_i that satisfies some clause of $C_{j,k}$. This means x_i makes the literal l_i true and l_i occurs in a clause of $C_{j,k}$. Therefore, for each k , at least $|\alpha(u_{j,k})| = |C_{j,k}|$ literals from the set $\{l_i \mid i \in J_j\}$ occur in clauses of $C_{j,k}$ and are made true by the assignment x .

Let $f : C_j \rightarrow \{0, 1, 2, 3\}$ be the function assigning to each clause $c \in C_j$ the number of literals of c in $\{l_i \mid i \in J_j\}$. By the above, $\sum_{c \in C_{j,k}} f(c) \geq |C_{j,k}|$ for $1 \leq k \leq 2n_{\mathcal{F}}$. Since each literal in $\{l_i \mid i \in J_j\}$ belongs to some clause of C_j , we have $\sum_{c \in C_j} f(c) = |J_j| = |C_j|$. Then,

$$\sum_{c \in C_{j,k}} f(c) = \sum_{c \in C_j} f(c) - \sum_{c \in C_{j,n_{\mathcal{F}}+k}} f(c) \leq |C_j| - |C_{j,n_{\mathcal{F}}+k}| = |C_{j,k}|.$$

Hence $\sum_{c \in C_{j,k}} f(c) = |C_{j,k}|$ for $1 \leq k \leq 2n_{\mathcal{F}}$. Let $g : C_j \rightarrow \{0, 1, 2, 3\}$ be the constant function $g \equiv 1$. Note that

$$\sum_{c \in C_{j,k}} g(c) = |C_{j,k}| = \sum_{c \in C_{j,k}} f(c).$$

Since $\{C_{j,1}, \dots, C_{j,n_{\mathcal{F}}}\}$ is a 4-detecting family, this implies that $f \equiv 1$. Thus, for every clause c of C_j we have $f(c) = 1$, meaning that there is a literal from the set $\{l_i \mid i \in J_j\}$ in this clause. All these literals are made positive by the assignment η , therefore all clauses of C_j are satisfied. Since $j = 1, \dots, n_C$ was arbitrary, this concludes the proof that η is a satisfying assignment for ϕ . \square

The construction can clearly be made in polynomial time and the total number of vertices is $n_V + n_C \cdot \mathcal{O}(\frac{b}{\log b}) + n_C = \mathcal{O}(\frac{n}{\log b})$. Moreover, we get a proper 3-coloring of G , by coloring vertices of the type v_i by color 1, vertices of the type $u_{j,k}$ by color 2, and vertices of the type w_j by color 3. By Lemmas 11 and 12, this concludes the proof of Theorem 9.

3.2 The uniform case

In this section we reduce the nonuniform case to the uniform one, and state the resulting lower bound on the complexity of LIST $(a:b)$ -COLORING.

Lemma 13. *For any instance $I = (G, \beta, L)$ of NONUNIFORM LIST $(a:b)$ -COLORING where the graph G is t -colorable, there is an equivalent instance (G, L') of LIST $((a + tb):b)$ -COLORING. Moreover, given a t -coloring of G the instance (G, L') can be constructed in time polynomial in $|I| + b$.*

Proof. Let $c : V(G) \rightarrow [t]$ be a t -coloring of G . For every vertex v , define a set of filling colors $F(v) = \{a + c(v)b + i : i = 0, \dots, b - |\beta(v)| - 1\}$ and put

$$L'(v) = L(v) \cup F(v).$$

Let $\alpha : V(G) \rightarrow 2^{[a]}$ be an L -($a:\beta$)-coloring of G . We define a coloring $\alpha' : V(G) \rightarrow 2^{[a+tb]}$ by setting $\alpha'(v) = \alpha(v) \cup F(v)$ for every vertex $v \in V(G)$. Observe that $\alpha'(v) \subseteq L'(v)$ and $|\alpha'(v)| = |\alpha(v)| + (b - |\beta(v)|) = b$. Since α was a proper L -($a:\beta$)-coloring, adjacent vertices can only share the filling colors. However, the lists of adjacent vertices have disjoint subsets of filling colors, since these vertices are colored differently by c . It follows that α' is an L' -($a:b$)-coloring of G .

Conversely, let $\alpha' : V(G) \rightarrow 2^{[a+tb]}$ be an L' -($a:b$)-coloring of G . For every vertex v , we have $|\alpha'(v) \cap [a]| = b - |\alpha'(v) \cap F(v)| \geq b - (b - |\beta(v)|) = |\beta(v)|$. Define $\alpha(v)$ to be any cardinality $\beta(v)$ subset of $\alpha'(v) \cap [a]$. It is immediate to check that α is an L -($a:\beta$)-coloring of G . \square

We are now ready to prove one of our main results.

Theorem 3. *If there is an algorithm for LIST $(a:b)$ -COLORING that runs in time $2^{o(\log b) \cdot n}$, then ETH fails. This holds even if the algorithm is only required to work on instances where $a = \Theta(b^2 \log b)$ and $b = \Theta(b(n))$ for an arbitrarily chosen polynomial-time computable function $b(n)$ such that $b(n) \in \omega(1)$ and $b(n) = \mathcal{O}(n/\log n)$.*

Proof. Let $b(n)$ be a function as in the statement. We can assume w.l.o.g. that $2 \leq b(n) \leq n/\log_2 n$, for otherwise we can replace $b(n)$ with a function $b'(n) = 2 + \lfloor b(n)/c \rfloor$ in the reasoning below, where c is a big enough constant; note that $b'(n) = \Theta(b(n))$. Fix a function $g(b) = o(\log b)$ and assume there is an algorithm \mathcal{A} for LIST $(a:b)$ -COLORING that runs in time $2^{g(b) \cdot n}$, whenever $b = \Theta(b(n))$. Consider an instance of (3,4)-SAT with n variables. Let $b = b(n)$. By Theorem 9 in $\text{poly}(n)$ time we get an equivalent instance (G, β, L) of NONUNIFORM LIST $(a:(2b))$ -COLORING such that $a = \Theta(b^2 \log b)$, $|V(G)| = \mathcal{O}(\frac{n}{\log b})$, and a 3-coloring of G . Next, by Lemma 13 in $\text{poly}(n)$ time we get an equivalent instance (G, L') of LIST $((a + 6b):(2b))$ -COLORING. Finally, we solve the instance (G, L') using algorithm \mathcal{A} . Since $b(n) = \omega(1)$, we have $g(b(n)) = o(\log(b(n)))$, and \mathcal{A} runs in time $2^{o(\log b(n)) \cdot |V(G)|}$. Thus, we have solved the instance ϕ of (3,4)-SAT in time $2^{o(\log b(n)) \cdot |V(G)|} = 2^{o(\log b(n)) \cdot \frac{n}{\log b(n)}} = 2^{o(n)}$. By Corollary 6, this contradicts ETH. \square

4 From LIST $(a:b)$ -COLORING to $(a:b)$ -COLORING

In this section we reduce LIST $(a:b)$ -COLORING to $(a:b)$ -COLORING. This is done by adding a Kneser graph, and replacing the lists by edges to appropriate vertices of the Kneser graph. We will need the following well-known property of Kneser graphs (see e.g., Theorem 7.9.1 in the textbook [15]).

Theorem 14. *If $p > 2q$ then every homomorphism from $KG_{p,q}$ to $KG_{p,q}$ is an automorphism.*

We proceed with the reduction.

Lemma 15. *For any given instance of LIST $(a:b)$ -COLORING with n vertices, there exists an equivalent instance of $((a+b):b)$ -COLORING with $n + \binom{a+b}{b}$ vertices. Moreover, it can be computed in $\text{poly}(n, \binom{a+b}{b})$ -time.*

Proof. Let (G, L) be an instance of LIST $(a:b)$ -COLORING where G is a graph and $L: V(G) \rightarrow 2^{[a]}$ describes the lists of allowed colors. Define a graph K with $V(K) = \binom{[a+b]}{b}$ and

$$E(K) = \{XY : X, Y \in V(K) \text{ and } X \cap Y = \emptyset\}.$$

That is, K is isomorphic to the Kneser graph $KG_{a+b,b}$. Then let $V' = V(G) \uplus V(K)$ and

$$E' = E(G) \uplus E(K) \uplus \{vX : v \in V(G) \text{ and } X \in V(K) \text{ and } L(v) \cap X = \emptyset\}.$$

The graph $G' = (V', E')$ has $n + \binom{a+b}{b}$ vertices, and the construction can be done in time polynomial in $n + \binom{a+b}{b}$. Let G' be our output instance of $((a+b):b)$ -COLORING. We will show that it is equivalent to the instance (G, L) of LIST $(a:b)$ -COLORING.

Let us assume that $\alpha: V(G) \rightarrow \binom{[a]}{b}$ is an L -($a:b$)-coloring of G . Consider $\alpha': V(G') \rightarrow \binom{[a+b]}{b}$ such that

$$\alpha'(v) = \begin{cases} \alpha(v) & \text{for } v \in V(G) \\ v & \text{for } v \in V(K) = \binom{[a+b]}{b}. \end{cases}$$

We claim that α' is an $((a+b):b)$ -coloring of G' . Indeed, for every edge $uv \in E(G)$ we have $\alpha'(u) \cap \alpha'(v) = \alpha(u) \cap \alpha(v) = \emptyset$ because α is an L -($a:b$)-coloring of G . For every edge $XY \in E(K)$ we have $\alpha'(X) \cap \alpha'(Y) = X \cap Y = \emptyset$. For every edge $vX \in E(V(G), V(K))$ we have $\alpha'(v) \cap \alpha'(X) = \alpha(v) \cap X \subseteq L(v) \cap X = \emptyset$.

Now, let us assume that $\alpha': V(G') \rightarrow \binom{[a+b]}{b}$ is an $((a+b):b)$ -coloring of G' . Recall that α' is a homomorphism of G' to $KG_{a+b,b}$. Denote $\phi = \alpha'|_{V(K)}$. By Theorem 14, ϕ is an automorphism of K .

Define $\alpha'' = \phi^{-1} \circ \alpha'$. Then α'' is an $((a+b):b)$ -coloring of G' with the property that $\alpha''(X) = X$ for every $X \in V(K)$. We claim that $\alpha''|_{V(G)}$ is an L -($a:b$)-coloring of G . Since α'' is a $((a+b):b)$ -coloring of G' , it suffices to show that $\alpha''(v) \subseteq L(v)$ for every vertex $v \in V(G)$. Pick a color $\gamma \notin L(v)$. Let X_γ be the b -element set consisting of γ and arbitrary $b-1$ elements from $[a+b] \setminus ([a] \cup \{\gamma\})$. Then $L(v) \cap X_\gamma = \emptyset$ and hence $vX_\gamma \in E(G')$. It follows that $X_\gamma \cap \alpha''(v) = \alpha''(X_\gamma) \cap \alpha''(v) = \emptyset$, and in particular $\gamma \notin \alpha''(v)$. Thus, $\alpha''(v) \subseteq L(v)$ as required. \square

We now prove our main result.

Theorem 1. *If there is an algorithm for $(a:b)$ -COLORING that runs in time $f(b) \cdot 2^{o(\log b) \cdot n}$, for some computable function $f(b)$, then ETH fails. This holds even if the algorithm is only required to work on instances where $a = \Theta(b^2 \log b)$.*

Proof. Fix a computable function $f(b)$, a function $g(b) = o(\log b)$ and assume there is an algorithm \mathcal{A} for $(a:b)$ -COLORING that runs in time $f(b) \cdot 2^{g(b) \cdot n}$ for a given n -vertex graph, whenever $a = \Theta(b^2 \log b)$. Without loss of generality we can replace $f(b)$ by any non-decreasing function $f'(n)$ such that $f'(n) \geq f(n)$ and $f'(n) > n$. Intuitively, we now define an unbounded function $b(N)$ which should be at least 2, at most the inverse of f , and small enough so that $2^{\mathcal{O}(b \log b)} \leq \frac{N}{\log b}$. The following function is $\omega(1)$ and a standard argument shows how to compute it in $\text{poly}(N)$ time (see Lemmas 3.2 and 3.4 in [4]).

$$b(N) = \min(\max\{b : f(b) \leq N\}, \max\{b : b \log b \leq \log N / \log \log N\}) + 2.$$

Consider an instance of (3,4)-SAT with N variables. Let $b = b(N)$. By Theorem 9 in $\text{poly}(N)$ time we get an equivalent instance (G, β, L) of NONUNIFORM LIST $(a:(2b))$ -COLORING such that $a = \Theta(b^2 \log b)$, $|V(G)| = \mathcal{O}(\frac{N}{\log b})$, and a 3-coloring of G . Next, by Lemma 13 in $\text{poly}(N)$ time we get an equivalent instance (G, L') of LIST $((a+6b):(2b))$ -COLORING. Then, by Lemma 15, in time $\text{poly}(N, \binom{a+8b}{2b}) = \text{poly}(N)$ we get an equivalent instance G' of $((a+8b):(2b))$ -COLORING such that $|V(G')| = |V(G)| + \binom{a+8b}{2b}$. Observe that since $a = \Theta(b^2 \log b)$ and $b \log b \leq \log N / \log \log N$,

$$\binom{a+8b}{2b} \leq (a+8b)^{2b} = 2^{\mathcal{O}(b \log b)} = 2^{\mathcal{O}(\log N / \log \log N)} = N^{o(1)} = o(N / \log b(N))$$

Hence $|V(G')| = \mathcal{O}(\frac{N}{\log b})$. Finally, we solve the instance G' using algorithm \mathcal{A} . Since $b(N) = \omega(1)$, we have $g(b(N)) = o(\log(b(N)))$. Therefore, \mathcal{A} runs in time

$$f(b) \cdot 2^{o(\log b(N)) \cdot |V(G')|} \leq N \cdot 2^{o(\log b(N)) \cdot \mathcal{O}(N / \log b(N))} = 2^{o(N)}$$

solving the instance ϕ of (3,4)-SAT in time $2^{o(N)}$. By Corollary 6, this contradicts ETH. \square

Corollary 2. *If there is an algorithm for GRAPH HOMOMORPHISM that runs in time $f(h) \cdot 2^{o(\log \log h) \cdot n}$, for some computable function $f(h)$, then ETH fails. This holds even if the algorithm is only required to work on instances where H is a Kneser graph $KG_{a,b}$ with $a = \Theta(b^2 \log b)$.*

Proof. Fix a computable function $f(h)$ and assume there is an algorithm \mathcal{A} for GRAPH HOMOMORPHISM that runs in time $f(h) \cdot 2^{o(\log \log h) \cdot n}$ for a given n -vertex graph, whenever H is a Kneser graph $K_{a,b}$ with $a = \Theta(b^2 \log b)$. Consider an instance of $(a:b)$ -COLORING with n vertices and $a = \Theta(b^2 \log b)$. This is an instance of GRAPH HOMOMORPHISM with $h = \binom{a}{b} \leq a^b = 2^{\mathcal{O}(b \log b)}$, hence \mathcal{A} solves it in

$$f(h) \cdot 2^{o(\log \log h) \cdot n} = f(2^{\mathcal{O}(b \log b)}) \cdot 2^{o(\log(b \log b)) \cdot n} \leq f'(b) \cdot 2^{o(\log b) \cdot n}$$

for some computable function $f'(b) \geq f(2^{\Theta(b \log b)})$, which contradicts Theorem 1. \square

5 Low-degree testing

In this section we derive lower bounds for (r, k) -MONOMIAL TESTING. In this problem, we are given an arithmetic circuit C over some field \mathbb{F} ; such a circuit may contain input, constant, addition, negation, multiplication, and inversion gates. One gate is designated to be the output gate, and it computes some polynomial P of the variables x_1, x_2, \dots, x_n that appear in the input gates. We assume that P is a homogenous polynomial of degree k , i.e., all its monomials have total degree k . The task is to verify whether P contains an r -monomial, i.e., a monomial in which every variable has its individual degree bounded by r , for a given parameter $r \leq k$. Abasi et al. [1] gave a very fast randomized algorithm for (r, k) -MONOMIAL TESTING.

Theorem 16 (Abasi et al. [1]). *Fix integers r, k with $2 \leq r \leq k$. Let $p \leq 2r^2 + 2r$ be a prime, and let $g \in \text{GF}(p)[x_1, \dots, x_n]$ be a homogenous polynomial of degree k , computable by a circuit C . Then, there is a randomized algorithm running in time $\mathcal{O}(r^{2k/r}|C|(rn)^{\mathcal{O}(1)})$ which*

- *with probability at least $1/2$ answers YES when g contains an r -monomial,*
- *always answers NO when g contains no r -monomial.*

This result was later derandomized by Gabizon et al. [14] under the assumption that the circuit is *non-cancelling*, that is, it contains only input, addition, and multiplication gates. Many concrete problems like r -SIMPLE k -PATH can be reduced to (r, k) -MONOMIAL TESTING by encoding the set of candidate objects as monomials of some large polynomial, so that “good” objects correspond to monomials with low individual degrees. As we will see in a moment, this is also the case for LIST $(a:b)$ -COLORING.

Let $(G = (V, E), L)$ be an instance of the LIST $(a:b)$ -COLORING problem and let \mathcal{I} be the family of all independent sets of G . We denote $n = |V|$. Let $C_a(G, L)$ denote the set of all functions $c : V \rightarrow 2^{[a]}$ such that for every edge $uv \in E$ the sets $c(u)$ and $c(v)$ are disjoint, and for every vertex v we have $c(v) \subseteq L(v)$. Consider the following polynomial in $n(a+1)$ variables $\{x_v\}_{v \in V}$ and $\{y_{v,j}\}_{v \in V, j \in [a]}$, over $\text{GF}(2)$.

$$p_G = \sum_{\substack{c \in C_a(G, L) \\ \sum_v |c(v)| = bn}} \prod_{v \in V} x_v^{|c(v)|} \prod_{j \in c(v)} y_{v,j}. \quad (1)$$

Note that every summand in expression (1) has a different set of variables, therefore it corresponds to a monomial (with coefficient 1). Then the following proposition is immediate.

Proposition 17. *There is a list $(a:b)$ -coloring of graph G iff p_G contains a b -monomial.*

Now we show that p_G can be evaluated relatively fast.

Lemma 18. *The polynomial p_G can be evaluated using a circuit of size $2^n \text{poly}(a, n)$, which can be constructed in time $2^n \text{poly}(a, n)$*

Proof. Consider the following polynomial:

$$q_G = \prod_{j=1}^a \sum_{I \in \mathcal{I}} \prod_{v \in I} x_v y_{v,j}. \quad (2)$$

Observe that p_G is obtained from q_G by removing all monomials of degree different than $2bn$. Eq. (2) shows that q_G can be evaluated by a circuit C_q of size $|\mathcal{I}| \text{poly}(a, n) \leq 2^n \text{poly}(a, n)$, which can be

constructed in time $2^n \text{poly}(a, n)$. We obtain from C_q a circuit C_p for p_G by splitting gates according to degrees, in a bottom-up fashion, as follows.

Every input gate u of C_q is replaced with a gate u_1 in C_p . Every addition gate u with inputs x and y in C_q is replaced in C_p by $2an$ addition gates u_1, \dots, u_{2an} , where u_i has inputs x_i and y_i (whenever x_i and y_i exist). Every multiplication gate u with inputs x and y in C_q is replaced in C_p by $2an$ addition gates u_1, \dots, u_{2an} . Moreover, for every pair of integers $1 \leq r, s \leq 2an$ we create a multiplication gate $u_{r,s}$ with inputs x_r and y_s (whenever they exist) and make it an input of the addition gate u_{r+s} . It is easy to see that for every gate u of C_q , for every i , the gate u_i of C_p evaluates the same polynomial as u , but restricted to monomials in which the total degree is equal to i . When o is the output gate of C_q , then o_{2bn} is the output gate of C_p . Clearly, $|C_p| \leq (2an + 1)^2 |C_q|$, and C_p can be constructed from C_q in time $2^n \text{poly}(a, n)$. \square

Since p_G is a homogenous polynomial of degree $k = 2bn$, by putting $r = b$ we can combine Proposition 17, Theorem 16 and Lemma 18 to get yet another polynomial-space algorithm for LIST $(a:b)$ -COLORING, running in time $b^{\mathcal{O}(n)} \cdot \text{poly}(n)$. Similarly, if the running time in Theorem 16 was improved from $2^{o(\log r/r) \cdot k} \cdot |C| \text{poly}(r, n)$, then we would get an algorithm for LIST $(a:b)$ -COLORING in time $2^{o(\log b) \cdot n} \cdot \text{poly}(n)$, which contradicts ETH by Theorem 3. However, a careful examination shows that this chain of reductions would only yield instances of (r, k) -MONOMIAL TESTING with $r = \mathcal{O}(\sqrt{k/\log k})$. Hence, this does not exclude the existence of a fast algorithm that works only for large r . Below we show a more direct reduction, which excludes fast algorithms for a wider spectrum of pairs (r, k) .

In the CARRY-LESS SUBSET SUM problem, we are given $n + 1$ numbers s, a_1, \dots, a_n , each represented as n decimal digits. For any number x , the j -th decimal digit of x is denoted by $x^{(j)}$. It is assumed that $\sum_{i=1}^n a_i^{(j)} < 10$, for every $j = 1, \dots, n$. The goal is to verify whether there is a sequence of indices $1 \leq i_1 < \dots < i_k \leq n$ such that $\sum_{q=1}^k a_{i_q} = s$. (Note that by the small sum assumption, this is equivalent to the statement that $\sum_{q=1}^k a_{i_q}^{(j)} = s^{(j)}$, for every $j = 1, \dots, n$). The standard NP-hardness reduction from 3-SAT to SUBSET SUM in fact outputs an instance of CARRY-LESS SUBSET SUM of linear size, yielding the following.

Lemma 19. *Unless ETH fails, there is no algorithm that solves CARRY-LESS SUBSET SUM with n numbers in time $2^{o(n)}$.*

Proof. Let φ be an instance of 3-SAT with N variables and M clauses. By a standard NP-hardness reduction for SUBSET SUM (see e.g. the textbook of Cormen et al. [8]) in polynomial time one can build an equivalent instance of CARRY-LESS SUBSET SUM, with $\mathcal{O}(N + M)$ numbers, each having of $\mathcal{O}(N + M)$ decimal digits, and with the sum of j -th digit in all the numbers not exceeding 7. In case the number of numbers is different from the length of their decimal representations, we can make them equal by padding the instance by zero numbers or with zeroes in the decimal representations. Thus, by Theorem 4, an $2^{o(n)}$ algorithm for CARRY-LESS SUBSET SUM would contradict ETH. \square

We proceed to reducing CARRY-LESS SUBSET SUM to (r, k) -MONOMIAL TESTING. Let us choose a parameter $t \in \{1, \dots, n\}$. We assume w.l.o.g. that $n \bmod t = 0$, for otherwise we add $t - (n \bmod t)$ zeroes at the end of every input number. Let $q = n/t$. For an n -digit decimal number x , for every $j = 1, \dots, t$, let $x^{[j]}$ denote the q -digit number given by the j -th block of q digits in x , i.e.,

$$x^{[j]} = (x^{(jq-1)} \dots x^{((j-1)q)})_{10}.$$

Let $r = 10^q - 1$. Define the following polynomial over $\text{GF}(2)$:

$$q_S = \prod_{i=1}^n \left(y_i + z_i \cdot \prod_{j=1}^t x_j^{a_i^{[j]}} \right) \cdot \prod_{j=1}^t x_j^{r-s^{[j]}}.$$

Proposition 20. (s, a_1, \dots, a_n) is a YES-instance of CARRY-LESS SUBSET SUM iff q_S contains the monomial $\prod_{j=1}^t x_j^r \prod_{i \notin S} y_i \prod_{i \in S} z_i$, for some $S \subseteq \{1, \dots, n\}$. \square

Proof. Consider the following polynomial over $\text{GF}(2)$:

$$r_S = \sum_{S \subseteq \{1, \dots, n\}} \prod_{j=1}^t x_j^{\sum_{i \in S} a_i^{[j]} + r - s^{[j]}} \prod_{i \notin S} y_i \prod_{i \in S} z_i.$$

The summands in the expression above have unique sets of y_i variables, so each of them corresponds to a monomial (of coefficient 1). It is clear that these monomials where for every j the degree of x_j is exactly r are in one-to-one correspondence with solutions of the instance (s, a_1, \dots, a_n) . The claim follows by observing that polynomials r_S and q_S coincide. \square

Let p_S denote the polynomial obtained from q_S by filtering out all the monomials of degree different than $k = tr + n$.

Proposition 21. (s, a_1, \dots, a_n) is a YES-instance of CARRY-LESS SUBSET SUM iff p_S contains an r -monomial.

Proof. If (s, a_1, \dots, a_n) is a YES-instance and let then by Proposition 20 polynomial q_S contains the monomial $\prod_{j=1}^t x_j^r \prod_{i \notin S} y_i \prod_{i \in S} z_i$, which is an r -monomial. This monomial has degree $tr + n$, so it is contained in p_S as well.

Conversely, assume p_S contains an r -monomial m . Every monomial of q_S (and hence also of p_S) contains exactly one of the variables y_i and z_i , with degree 1, for every $i = 1, \dots, n$. It means that the total degree of x_j -type variables in m is tr . Hence, since m is an r -monomial, each of x_j 's has degree exactly r . In other words, m is of the form $\prod_{j=1}^t x_j^r \prod_{i \notin S} y_i \prod_{i \in S} z_i$, for some $S \subseteq \{1, \dots, n\}$. Then (s, a_1, \dots, a_n) is a YES-instance of CARRY-LESS SUBSET SUM by Proposition 20. \square

Proposition 22. p_S can be evaluated by a circuit of size $\mathcal{O}(nt^2r + n^2t)$, which can be constructed in time polynomial in $n + t + r$.

Proof. Polynomial q_S can be evaluated by a circuit of size $\mathcal{O}(nt)$. The circuit for p_S is built using the construction from Lemma 18. Thus, its size is $\mathcal{O}(nt(tr + n)) = \mathcal{O}(nt^2r + n^2t)$. \square

We are ready to give our main lower bound for (r, k) -MONOMIAL TESTING. We state it in the most general form, which unfortunately is also quite technical. Next, we derive an exemplary corollary that gives a lower bound for r expressed as a function of k .

Theorem 23. If there is an algorithm solving (r, k) -MONOMIAL TESTING in time $2^{o(k \log r/r)} |C|^{\mathcal{O}(1)}$, then ETH fails. The statement remains true even if the algorithm works only for instances where $r = 2^{\Theta(n/t(n))}$ and $k = t(n)2^{\Theta(n/t(n))}$, for an arbitrarily chosen function $t : \mathbb{N} \rightarrow \mathbb{N}$ computable in $2^{o(n)}$ time, such that $t(n) = \omega(1)$ and $t(n) \leq n$ for every n .

Proof. By Lemma 19, it suffices to give an algorithm for CARRY-LESS SUBSET SUM that works in time $2^{o(n)}$, where n is the number of input numbers. Let $t = t(n)$ and $q = n/t$, $r = 10^q - 1$, $k = tr + n$ as before. Note that $r = 2^{\Theta(n/t(n))}$. Also, since $10^{n/t(n)} = \Omega(n/t(n))$, $k = \Theta(t(n)10^{n/t(n)} + n) = \Theta(t(n)10^{n/t(n)}) = t(n)2^{\Theta(n/t(n))}$.

By Proposition 21, solving CARRY-LESS SUBSET SUM is equivalent to detecting an r -monomial in p_S , which is a homogenous polynomial of degree $k = tr + n$. Let C be the circuit for p_S ; by Proposition 22 we have $|C| = \mathcal{O}(nt^2r + n^2t)$. If this can be done in time $2^{o(k \log r/r)}|C|^{\mathcal{O}(1)}$, we get an algorithm for CARRY-LESS SUBSET SUM running in time

$$2^{o(k \log r/r)}|C|^{\mathcal{O}(1)} = 2^{o((tr+n)q/r)}(ntr)^{\mathcal{O}(1)} = 2^{o(n+nq/10^q)}(ntr)^{\mathcal{O}(1)} = 2^{o(n)}(ntr)^{\mathcal{O}(1)}.$$

Recall that $t \leq n$ and $r = 10^{n/t} - 1 = 2^{o(n)}$, since $t = t(n) = \omega(1)$. Hence $(ntr)^{\mathcal{O}(1)} = 2^{o(n)} \text{poly}(n)$. The claim follows. \square

Theorem 24. *Let $\sigma \in [0, 1)$. Then, unless ETH fails, there is no algorithm for (r, k) -MONOMIAL TESTING that solves instances with $r = \Theta(k^\sigma)$ in time $2^{o(k \cdot \frac{\log r}{r})} \cdot |C|^{\mathcal{O}(1)}$.*

Proof. We prove that an algorithm for (r, k) -MONOMIAL TESTING with properties as in the statement can be used to derive an algorithm for the same problem with properties as in the statement of Theorem 23, which implies that ETH fails. Take t to be a positive integer not larger than n such that

$$\frac{1}{2} \leq \frac{10^{n/t} - 1}{(t \cdot (10^{n/t} - 1) + n)^\sigma} \leq 2; \quad (3)$$

it can be easily verified that since $\sigma < 1$, for large enough n such an integer $t \leq n$ always exists. Moreover, we have that $t = t(n) \in \omega(1)$ and $t(n)$ can be computed in polynomial time by brute-force. Hence, $t(n)$ satisfies the properties stated in Theorem 23.

Let $t = t(n)$ and $q = n/t$. Define $r = 10^q - 1$ and $k = tr + n$, then (3) is equivalent to

$$1/2 \leq r/k^\sigma \leq 2.$$

Hence $r = \Theta(k^\sigma)$. Consequently, the assumed algorithm solves (r, k) -MONOMIAL TESTING in time $2^{o(k \log r/r)}|C|^{\mathcal{O}(1)}$, however in the proof of Theorem 23 we have shown that the existence of an algorithm that achieves such a running time for this particular choice of parameters implies that ETH fails. \square

Note that Theorem 24 in particular implies that (r, k) -MONOMIAL TESTING does not admit an algorithm that achieves running time $2^{o(\frac{\log r}{r}) \cdot k} \cdot |C|^{\mathcal{O}(1)}$ for any given r .

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